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Poisson equations derived from certain H-J-B equations of ergodic type

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1 Introduction

In studying problems of large time asymptotics of the probability minimizing down-side risk, which arise from mathematical finance, we discussed duality relation between the minimizing probability on long term and risk-sensitive sensitive asset allocatiion on infinite time horizon. As a result we get the limit value of the minimizing probability as the Legendre transformation of the value of risk-sensitive stochastic control on infinite time horizon along the line of the idea of large deviation principle. Seeking the probability minimizing such down-side risk on long term is a non standard stochastic control problem and it is not directly obtained. In proving the duality relation key analysis lies in the studies of Poisson equations derived from H-J-B equations of ergodic type corresponding to the risk-sensitive stochastic control as their derivatives. In this article we present the results on the large time asymptotics of the probability and then state the results concerning analysis of the Poisson equations. Full papers will be seen elsewhere.

2 Results about problems of large time asymptotics arising from mathematical finance

Consider a market model with $m+1$ securities and n factors, where the bond price is governed by ordinary differential equation

$$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0. \tag{2.1}$$

The other secutity prices and factors are assumed to satisfy stochastic differential equations

$$\begin{aligned} &dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \textstyle\sum_{k=1}^{n+m} \sigma^i_k(X_t)dW^k_t\}, \\ &S^i(0) = s^i, \ \ i = 1, ..., m \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \\ &X(0) = x, \end{aligned} \tag{2.3}$$

where $W_t = (W_t^k)_{k=1,..,(n+m)}$ is an $m+n-$ dimensional standard Brownian motion process on a probability space $(\Omega, \mathcal{F}, P)$. Let N_t^i be the number of the shares of i -th security. Then the total wealth the investor possesses is defined as

$$V_t = \sum_{i=0}^{m} N_t^i S_t^i$$

the portfolio proprtion invested to i-th security as

$$h_t^i = \frac{N_t^i S_t^i}{V_t}, \quad i = 0, 1, 2, \dots, m$$

$N_t = (N_t^0, N_t^1, N_t^2, \dots, N_t^m)$ (or, $h_t = (h_t^1, \dots, h_t^m)$) is called self-financing if

$$dV_t = \sum_{i=0}^{m} N_t^i dS_t^i = \sum_{i=0}^{m} \frac{V_t h_t^i}{S_t^i} dS_t^i$$

and it means

$$\begin{aligned} \tfrac{dV_t}{V_t} &= h_t^0 r(X_t)dt + \textstyle\sum_{i=1}^m h_t^i\{\alpha^i(X_t)dt + \sum_{j=1}^{n+m} \sigma_j^i(X_t)dW_t^j\} \\ &= r(X_t)dt + \textstyle\sum_{i=1}^m h_t^i\{(\alpha^i(X_t) - r(X_t))dt + \sum_{j=1}^{n+m} \sigma_j^i(X_t)dW_t^j\} \end{aligned}$$

Here we note that h_t is defined as m-vector consisting of $h_t^1, \dots, h_t^m$ since $h_t^0 = 1 - \sum_{i=1}^m h_t^i$ holds by definition.

As for filtration to be satisfied by admissible investment strategies

$$\mathcal{G}_t = \sigma(S(u), X(u), \quad u \leq t)$$

is relevant in the present problem and we introduce the following definition.

Definition 2.1 *$h(t)_{0\leq t\leq T}$ is said an invetment strategy if $h(t)$ is an R^m valued $\mathcal{G}_t$- progressively measurable stochastic process such that*

$$P\Big(\int_0^T |h(s)|^2 ds < \infty, \quad \forall T\Big) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. For given $h \in \mathcal{H}(T)$ the process $V_t = V_t(h)$ representing the total wealth of the investor at time t is determined by the stochastic differential equation as was seen above:

$$\begin{aligned} \frac{dV_t}{V_t} &= r(X_t)dt + h(t)^*(\alpha(X_t) - r(X_t)\mathbf{1})dt + h(t)^*\sigma(X_t)dW_t, \\ V_0 &= v_0, \end{aligned} \tag{2.5}$$

where $\mathbf{1} = (1, 1, ..., 1)^*$.

We are interested in asymptotics of the probability minimizing a down-side risk against holding whole portfolio for the riskless security as the bench mark for a given constant κ:

$$J(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}(T)} \log P\Big(\frac{1}{T} \log \frac{V_T(h)}{S_T^0} \leq \kappa\Big). \tag{2.6}$$

If we take a strategy $h_t^0 \equiv 1$, then $V_T(h) = S_T^0$. Therefore, in considering (2.6) we are seeing how we could improve the down-side risk probability comparing with such trivial strategy on long term. We also study down-side risk minimization with the bench mark S^0 on infinite time horizon

$$(2.7) \qquad J_\infty(\kappa) := \inf_{h\in\mathcal{H}} \varliminf_{T\to\infty} \frac{1}{T}\log P(\frac{1}{T}\log\frac{V_T(h)}{S_T^0} \le \kappa).$$

The former will be shown related to the following risk-sensitive asset allocation problem with bench mark S^0. Namely, for a given constant $\gamma < 0$ consider the following asymptotics

$$(2.8) \qquad \hat{\chi}(\gamma) = \varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} J(v,x;h;T),$$

where

$$(2.9) \qquad J(v,x;h;T) = \log E[(\frac{V_T(h)}{S_T^0})^\gamma] = \log E[e^{\gamma\log(\frac{V_T(h)}{S_T^0})}],$$

and h ranges over the set $\mathcal{A}(T)$ of all addmissible investment strategies defined by

$$\mathcal{A}(T) = \{h \in \mathcal{H}(T); E[e^{\gamma\int_0^T h_s^*\sigma(X_s)dW_s - \frac{\gamma^2}{2}\int_0^T h_s^*\sigma\sigma^*(X_s)h_s ds}] = 1.$$

Then, we shall see that (2.6) could be considered as the dual problem to (2.8). While, the latter (2.7) is considered to corresponds to risk-sensitive asset allocation on infinite time horizon:

$$(2.10) \qquad \chi_\infty(\gamma) = \inf_{h\in\mathcal{A}} \varliminf_{T\to\infty} \frac{1}{T} J(v,x;h;T),$$

where

$$\mathcal{A} = \{h; h \in \mathcal{A}(T); \forall T\}.$$

We shall consider these problems under the assumptions that

$$(2.11) \qquad \lambda,\ \beta,\ \sigma,\ \alpha \text{ and } r \text{ are globally Lipshitz, smooth}$$

and

$$(2.12) \qquad \begin{cases} c_1|\xi|^2 \le \xi^*\lambda\lambda^*(x)\xi \le c_2|\xi|^2, & c_1,\ c_2 > 0,\ \xi \in R^n, \\ c_1|\zeta|^2 \le \zeta^*\sigma\sigma^*(x)\zeta \le c_2|\zeta|^2, & \zeta \in R^m \end{cases}$$

hold. In considering these problems we first introduce value function

$$(2.13) \qquad v(t,x) = \inf_{h.\in\mathcal{A}(T-t)} \log E[e^{\gamma\log(\frac{V_{T-t}(h)}{S_{T-t}^0})}]$$

Note that

$$e^{\gamma\log V_T} = v_0^\gamma e^{\gamma\int_0^T\{r(X_s)+h_s^*\hat{\alpha}(X_s)-\frac{1}{2}h_s^*\sigma\sigma^*(X_s)h_s\}ds+\gamma\int_0^T h_s^*\sigma(X_s)dW_s}$$

where $\hat{\alpha}(x) = \alpha(x) - r(x)\mathbf{1}$. Therefore

$$e^{\gamma(\log V_T - \log S_T^0)} = v_0^\gamma e^{\gamma\int_0^T \eta(X_s,h_s)ds + \gamma\int_0^T h_s^*\sigma(X_s)dW_s - \frac{\gamma^2}{2}\int_0^T h_s^*\sigma\sigma^*(X_s)h_s ds},$$

where

$$\eta(x,h) = h^*\hat{\alpha}(x) - \frac{1-\gamma}{2}h^*\sigma\sigma^*(x)h.$$

Thus, by introducing a probability measure

$$P^h(A) = E[e^{\gamma\int_0^T h_s^*\sigma(X_s)dW_s - \frac{\gamma^2}{2}\int_0^T h_s^*\sigma\sigma^*(X_s)h_s ds} : A]$$

the dynamics of the factor process can be written as

$$dX_t = \{\beta(X_t) + \gamma\lambda\sigma^*(X_t)h_t\}dt + \lambda(X_t)dW_t^h, \quad X_0 = x$$

with new Brownian motion process W_t^h defined by

$$W_t^h := W_t - \gamma\int_0^t \sigma^*(X_s)h_s ds$$

and so the value function is written as

(2.14)
$$v(t,x) = \gamma\log v_0 + \inf_{h.\in\mathcal{A}(T)} \log E^h[e^{\gamma\int_0^{T-t}\eta(X_s,h_s)ds}]$$

The H-J-B equation for the value function $v(t,x)$ is

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \frac{1}{2}(Dv)^*\lambda\lambda^* Dv + \inf_h\{[\beta+\gamma\lambda\sigma^* h]^* Dv + \gamma\eta(x,h)\} = 0, \\ v(T,x) = \gamma\log v_0 \end{cases}$$

which is also written as

(2.15)
$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\gamma^* Dv + \frac{1}{2}(Dv)^*\lambda N_\gamma^{-1}\lambda^* Dv - U_\gamma = 0, \\ v(t,x) = \gamma\log v_0 \end{cases}$$

where

$$\begin{aligned} \beta_\gamma &= \beta + \tfrac{\gamma}{1-\gamma}\lambda\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha} \\ N_\gamma^{-1} &= I + \tfrac{\gamma}{1-\gamma}\sigma^*(\sigma\sigma^*)^{-1}\sigma \\ U_\gamma &= -\tfrac{\gamma}{2(1-\gamma)}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} \end{aligned}$$

Remark 2.1

$$\inf_{h\in R^m}\{[\gamma\lambda\sigma^* h]^* Dv - \gamma(1-\gamma)2h^*\sigma\sigma^* h + \gamma h^*\hat{\alpha}\}$$

$$= \inf_{h\in R^m}\{-\tfrac{\gamma(1-\gamma)}{2}[h - \tfrac{1}{1-\gamma}(\sigma\sigma^*)^{-1}(\hat{\alpha}+\sigma\lambda^* Dv)]^*\sigma\sigma^*[h - \tfrac{1}{1-\gamma}(\sigma\sigma^*)^{-1}(\hat{\alpha}+\sigma\lambda^* Dv)]$$

$$+\tfrac{\gamma}{2(1-\gamma)}(\hat{\alpha}+\sigma\lambda^* Dv)^*(\sigma\sigma^*)^{-1}(\hat{\alpha}+\sigma\lambda^* Dv)\}$$

Therefore the function

$$\hat{h}(t,x) := \frac{1}{1-\gamma}(\sigma\sigma^*)^{-1}(\hat{\alpha}+\sigma\lambda^* Dv)$$

defines the generator of the optimal diffusion $\hat{L}$:

$$\hat{L}\psi := \frac{1}{2}tr[\lambda\lambda^* D^2\psi] + [\beta + \frac{\gamma}{1-\gamma}\lambda\sigma^*(\sigma\sigma^*)^{-1}(\hat{\alpha}+\sigma\lambda^* Dv)]^* D\psi$$

Remark 2.2 *The following notation is useful. Set* $\Sigma := (\sigma\sigma^*)^{-1}\sigma$. *Then,*

$$\Sigma^* = \sigma^*(\sigma\sigma^*)^{-1},\ \ \Sigma\Sigma^* = (\sigma\sigma^*)^{-1},\ \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma = \sigma^*(\sigma\sigma^*)^{-1}\sigma$$

Moreover, we see that

$$\Sigma N_\gamma^{-1} = \frac{1}{1-\gamma}\Sigma,\ \ N = I - \gamma\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma = I - \gamma\sigma^*(\sigma\sigma^*)^{-1}\sigma$$

Set $\bar{v} = -v$. Then,

$$(2.16)\qquad \begin{cases} \frac{\partial \bar{v}}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}] + \beta_\gamma^* D\bar{v} - \frac{1}{2}(D\bar{v})^*\lambda N_\gamma^{-1}\lambda^* D\bar{v} + U_\gamma = 0 \\ \bar{v}(T,x) = -\gamma\log v_0 \end{cases}$$

Since $I - \sigma^*(\sigma\sigma^*)^{-1}\sigma \geq 0$, which is easily seen by taking $\xi = \sigma^*\zeta + \mu$, with μ orthogonal to the range of σ^* and seeing that $\xi^*(I - \sigma^*(\sigma\sigma^*)^{-1}\sigma)\xi = \mu^*\mu$, we have

$$(2.17)\qquad \frac{1}{1-\gamma}I \leq N^{-1} \leq I$$

As for existence of the solution to (2.16) satisfying sufficient regularities we have the following results (cf. [3],[14]).

Theorem 2.1 ([3],[14]) *Assume (2.11) and (2.12). Then, H-J-B equation (2.16) has a solution such that*

$$\bar{v}(t,x) + \gamma\log v_0 \geq 0$$

$$\bar{v},\ \frac{\partial\bar{v}}{\partial t},\ \frac{\partial\bar{v}}{\partial x_k},\ \frac{\partial^2\bar{v}}{\partial x_k\partial x_j} \in L^p(0,T;L^p_{loc}(R^n)),\ 1 < \forall p < \infty$$

$$\frac{\partial^2\bar{v}}{\partial t^2},\ \frac{\partial^2\bar{v}}{\partial x_k\partial t},\ \frac{\partial^3\bar{v}}{\partial x_k\partial x_j\partial x_l},\ \frac{\partial^3\bar{v}}{\partial x_k\partial x_j\partial t} \in L^p(0,T;L^p_{loc}(R^n)),$$

$$\frac{\partial\bar{v}}{\partial t} \leq 0$$

and

$$|\nabla\bar{v}|^2 - c_0\frac{\partial\bar{v}}{\partial t} \leq C(|\nabla Q_\gamma|^2_{2\rho} + |Q_\gamma|^2_{2\rho} + |\nabla(\lambda\lambda^*)|^2_{2\rho}$$

$$+|\nabla\beta_\gamma|_{2\rho} + |\beta_\gamma|^2 + |U_\gamma|_{2\rho} + |\nabla U_\gamma|^2 + 1)$$

$x \in B_\rho$, $t \in [0,T)$, *where* $Q_\gamma = \lambda N_\gamma^{-1}\lambda^*$, $c_0 = \frac{4(1+c)(1-\gamma)}{-\gamma}$, $c > 0$, *and* C *is a universal constant*

For $\hat{h}(t,x)$ we consider stochastic differential equation

$$dX_t = \{\beta(X_t) + \gamma\lambda\sigma^*(X_t)\hat{h}(t,X_t)\}dt + \lambda(X_t)dW_t^{\hat{h}},\quad X_0 = x$$

and define $\hat{h}_t := \hat{h}(t,X_t)$ for the solution X_t of the stochastic differential equation. The following is a so called verification theorem the proof of which is seen in [14] Proposition 2.1.

Proposition 2.1 ([14]) *Assume (2.11) and (2.12). Then,* $\hat{h}_t^{(\gamma,T)} \equiv \hat{h}_t := \hat{h}(t, X_t) \in \mathcal{A}(T)$ *and it is optimal:*

$$v(0,x) = \inf_{h.} \log E[e^{\gamma(\log V_T(h) - \log S_T^0)}] = \log E[e^{\gamma(\log V_T(\hat{h}) - \log S_T^0)}] \tag{2.18}$$

Let us consider an H-J-B equation of ergodic type which is considered the limit equation of (2.15). Namely,

$$\chi = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta_\gamma^* Dw + \frac{1}{2}(Dw)^*\lambda N_\gamma^{-1}\lambda^* Dw - U_\gamma \tag{2.19}$$

Set

$$G(x) := \beta(x) - \lambda\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha}(x)$$

and assume that

$$G(x)^* x \leq -c_G|x|^2 + c'_G, \quad c_G,\ c'_G > 0 \tag{2.20}$$

$$\hat{\alpha}\Sigma\Sigma^*\hat{\alpha} \to \infty, \quad \text{as } |x| \to \infty \tag{2.21}$$

Under these assumptions we have a solution to the H-J-B equation of ergodic type.

Proposition 2.2 *Assume (2.11),(2.12), (2.20) and (2.21). Then (2.19) has a solution* (χ, w) *such that* $w \in C^2(R^n)$,

$$w(x) \to -\infty \ \text{ as } |x| \to \infty,$$

and such solution is unique up to additive constants with respect to w.

We furthermore assume that

$$\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} \geq c_0(1+|x|^2), \quad c_0 > 0 \tag{2.22}$$

Then we have the following theorem.

Theorem 2.2 *Under assumptions (2.11),(2.12),(2.20) and (2.22) we have*

$$\hat{\chi}(\gamma) = \lim_{T\to\infty} \frac{1}{T} v(0,x;T) = \chi(\gamma) \tag{2.23}$$

The following results are important to prove our main results.

Proposition 2.3 *Under the assumptions of Theorem 2.2* $\chi(\gamma)$ *is convex and differentiable. Furthermore*

$$\lim_{\gamma\to-\infty} \chi'(\gamma) = 0$$

Now we can state our main theorem.

Theorem 2.3 *Under the assumptions of Theorem 2.2 for* $0 < \kappa < \hat{\chi}'(0-)$

$$J(\kappa) = -\inf_{k\in(-\infty,\kappa]}\sup_{\gamma<0}\{\gamma k - \hat{\chi}(\gamma)\} = \inf_{\gamma<0}\{\hat{\chi}(\gamma) - \gamma\kappa\} \tag{2.24}$$

Moreover, for $\gamma(\kappa)$ *such that* $\hat{\chi}'(\gamma(\kappa)) = \kappa \in (0, \hat{\chi}'(0-))$ *take a strategy* $\hat{h}_t^{(\gamma(\kappa),T)}$ *defined in Proposition 2.1. Then,*

$$J(\kappa) = \lim_{T\to\infty}\frac{1}{T}\log P\Big(\frac{1}{T}\log\frac{V_T(\hat{h}^{(\gamma(\kappa),T)})}{S_T^0} \le \kappa\Big)$$

For $\kappa < 0$,

$$J(\kappa) = \inf_{\gamma<0}\{\hat{\chi}(\gamma) - \gamma\kappa\} = -\infty$$

For the solution $w = w^{(\gamma)}$ to H-J-B equation ergodic type (2.19) let us set

$$\bar{h}(x) = \frac{1}{1-\gamma}(\sigma\sigma^*)^{-1}(\hat{\alpha} + \sigma\lambda^* Dw)(x)$$

and consider stochastic differential equation

$$dX_t = \{\beta(X_t) + \gamma\lambda\sigma^*(X_t)\bar{h}(X_t)\}dt + \lambda(X_t)dW_t^{\bar{h}}, \quad X_0 = x \tag{2.25}$$

and define $\bar{h}_t^{(\gamma(\kappa))} := \bar{h}(X_t)$ for the solution X_t of the stochastic differential equation. Then we have the following Theorem.

Theorem 2.4 *Assume the assumptions of Theorem 2.2. Let* $0 < \kappa < \hat{\chi}'(0-)$ *and* $\gamma(\kappa)$ *be the same as above. We moreover assume that*

$$(Dw^{(\gamma)})^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^* Dw^{(\gamma)} < \hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha}, \quad \gamma = \gamma(\kappa) \tag{2.26}$$

Then,

$$J_\infty(\kappa) = J(\kappa) = -\inf_{k\in(-\infty,\kappa]}\sup_{\gamma<0}\{\gamma k - \hat{\chi}(\gamma)\} = \inf_{\gamma<0}\{\hat{\chi}(\gamma) - \gamma\kappa\}$$

and

$$J(\kappa) = \lim_{T\to\infty}\frac{1}{T}\log P\Big(\log\frac{V_T(h^{(\gamma(\kappa))})}{S_T^0} \le \kappa T\Big)$$

In the papers [7], [15] we have studied similar asymptotic behavior without bench mark case for linear Gaussian models in relation to asymptotics of the risk-sensitive portfolio optimization. Indeed, we have gotten duality relation between these problems and as a result an explicit expression of the limit value of the probability minimizing down-side risk for each case of full information and partial information. To get these results, key analysis has been in the studies of Poisson equations derived as the derivatives with respect to γ of the H-J-B equations of ergodic type corresponding to risk-sensitive control on infinite time horizon. Since the solutions of the H-J-B equations can be explicitly expresssed as the quadraric functions by using the solutions of Riccati equations for linear Gaussian models the analysis on differentiabilities of the solutions of the Riccati equations with respect to γ has been essential in these works.

In this article we treat general Markovian market models and discuss the duality relation between asymptotics of the probability minimizing down-side risk and risk-sensitive stochastic control. Since the solutions of H-J-B equations of ergodic type have not always explicit expressions we need to develop general discussions about differentiablities with respect to γ of H-J-B equation of ergodic type.

3 H-J-B equations of ergodic type

We shall study H-J-B equation of ergodic type:

$$-\chi = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{w}] + \beta_\gamma^* D\bar{w} - \frac{1}{2}(D\bar{w})^*\lambda N_\gamma^{-1}\lambda^* D\bar{w} + U_\gamma \tag{3.1}$$

Proposition 3.1 *Assume (2.11),(2.12), (2.20) and (2.21). Then (3.1) has a solution $(\chi, \bar{w})$ such that $\bar{w} \in C^2(R^n)$,*

$$\bar{w}(x) \to \infty \quad as \quad |x| \to \infty,$$

and such solution is unique up to additive constants with respect to $\bar{w}$.

To prove this proposition we first consider H-J-B equation of discounted type

$$\epsilon\bar{v}_\epsilon = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}_\epsilon] + \beta_\gamma^* D\bar{v}_\epsilon - \frac{1}{2}(D\bar{v}_\epsilon)^*\lambda N_\gamma^{-1}\lambda^* D\bar{v}_\epsilon + U_\gamma \tag{3.2}$$

Note that (3.2) can be written as

$$\epsilon\bar{v}_\epsilon = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}_\epsilon] + G^* D\bar{v}_\epsilon - \frac{1}{2}(\lambda D\bar{v}_\epsilon - \Sigma^*\hat{\alpha})^* N_\gamma^{-1}(\lambda^* D\bar{v}_\epsilon - \Sigma^*\hat{\alpha}) + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}. \tag{3.3}$$

Lemma 3.1 *Under the assumptions of Proposition 3.1 (3.2) has a solution $v_\epsilon \in C^2(R^n)$.*

Now let us consider linear equation

$$\epsilon\varphi_\epsilon = L\varphi_\epsilon + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}, \tag{3.4}$$

where

$$L\varphi = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\varphi] + G^* D\varphi$$

Set

$$\psi_\delta(x) := e^{\delta|x|^2}, \quad \delta > 0.$$

Then, by taking δ sufficiently small, we can see that there exists R_1 such that for $R > R_1$

$$L\psi_\delta(x) < -1, \quad \text{in } B_R^c.$$

Moreover, we see that L and ψ_δ satisfy assumption (7.3) in the last section. Set $K(x;\psi_\delta) = -L\psi_\delta$ and

$$F_\psi := \{u(x) \in W_{loc}^{2,p}(R^n);\ \sup_{x\in B_R^c} \frac{|u(x)|}{\psi_\delta(x)} < \infty\}$$

and

$$F_K := \{f(x) \in W_{loc}^{2,p}(R^n);\ \sup_{x\in B_R^c} \frac{|f(x)|}{-L\psi_\delta(x)} < \infty\}$$

Then, for $f \in F_K$ there exists a solution $\varphi \in F_\psi$ to

$$0 = L\varphi + f$$

if and only if

$$\int f(x)m(dx) = 0,$$

where $m(dx)$ is a invariant measure for L (cf. Proposition 7.4 in section 7). Therefore, setting

$$\chi_0 = \int \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}(x)m(dx),$$

there exists a solution $\varphi_0 \in F_\psi$ to

$$\chi_0 = L\varphi_0 + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}(x)$$

and it is known that $\epsilon\varphi_\epsilon$ converges to χ_0 as $\epsilon \to 0$ uniformly on each compact set.

In the following we shall assume

$$\hat{\alpha}\Sigma\Sigma^*\hat{\alpha} \geq c_0(|x|^2 + 1), \quad |x| \geq R \tag{3.5}$$

Then we have the following proposition.

Proposition 3.2 *Under the assumptions of Proposition 3.1 the solution $\bar{w}$ to (3.1) satisfies*

$$|\nabla\bar{w}(x)|^2 \leq c(|x|^2 + 1), \tag{3.6}$$

where c is a positive constant. If we moreover assume (3.5) then, for each $\gamma_0 < 0$ there exists a positive constant $c(\gamma_0)$ such that the nonnegative solution $\bar{w}(x;\gamma)$, $\gamma \leq \gamma_0$ satisfies

$$\bar{w}(x) \geq c(\gamma_0)|x|^2, \quad |x| \geq \exists R' \tag{3.7}$$

4 H-J-B equations and related stochastic control problems

Let us come back to H-J-B equation (2.16). According to assumption (2.12), we have a positive constant c_β such that

$$|\beta_\gamma(x)|^2 \leq c_\beta(|x|^2 + 1).$$

We strengthen condition (2.21) to (2.22). Then we have the following lemma.

Lemma 4.1 *Assume (2.11), (2.12) and (2.22) and $v_0 \geq 1$. Then, for each $t < T$ there exists a constant $k = k(T-t)$ such that*

$$\bar{v}(t, x; T) \geq k|x|^2 \tag{4.1}$$

Let us rewrite (2.16) as

$$\begin{cases} 0 = \frac{\partial\bar{v}}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}] + G^* D\bar{v} - \frac{1}{2}(\lambda D\bar{v} - \Sigma^*\hat{\alpha})^* N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha}) \\ \qquad\qquad\qquad\qquad\qquad + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}. \\ \bar{v}(T, x) = -\gamma \log v_0 \end{cases} \tag{4.2}$$

Noting that

$$-\tfrac{1}{2}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha})^* N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha})$$

$$= \inf_{z\in R^{n+m}}\{\tfrac{1}{2}z^* N_\gamma z - z^*\Sigma^*\hat{\alpha} + (\lambda z)^* D\bar{v}\}$$

$$= \inf_{z\in R^{n+m}}[\tfrac{1}{2}\{z + N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha}\}^* N_\gamma\{z + N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha}\}$$

$$-\tfrac{1}{2}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha})^* N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha})]$$

we can rewrite it again as

$$(4.3) \qquad \begin{cases} 0 = \frac{\partial\bar{v}}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}] + G^* D\bar{v} + \inf_{z\in R^{n+m}}\{(\lambda z)^* D\bar{v} + \varphi(x,z)\} \\ \bar{v}(T,x) = -\gamma\log v_0 \end{cases}$$

where

$$\varphi(x,z) = \frac{1}{2}z^* N_\gamma z - z^*\Sigma^*\hat{\alpha} + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}, \quad N_\gamma = I - \gamma\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma.$$

This H-J-B equation corresponds to the following stochastic control problem whose value is defined as

$$(4.4) \qquad \inf_{Z.\in\bar{\mathcal{A}}(T)} E[\int_0^T \varphi(Y_s, Z_s)ds - \gamma\log v_0],$$

where Y_t is a controlled process governed by stochastic differential equation

$$(4.5) \qquad dY_t = \lambda(Y_t)dW_t + \{G(Y_t) + \lambda(Y_t)Z_t\}dt, \quad Y_0 = x$$

with controlled process Z_t, which is an R^{n+m} valued progressively measurable process. To study this problem we introduce a value function for $0 \le t \le T$

$$v_*(t,x) = \inf_{Z.\in\bar{\mathcal{A}}(T-t)} E[\int_0^{T-t} \varphi(Y_s, Z_s)ds - \gamma\log v_0]$$

By the verification theorem the solution $\bar{v}$ to (4.3) can be identitied with the value function v_*. Moreover, set

$$\hat{z}(s,x) = -N_\gamma^{-1}(\lambda^* D\bar{v} - \Sigma^*\hat{\alpha})(s,x),$$

which attains the infimum in (4.3), and consider stochastic differential equation

$$(4.6) \qquad d\hat{Y}_t = \lambda(\hat{Y}_t)dW_t + \{G(\hat{Y}_t) + \lambda(\hat{Y}_t)\hat{Z}(t,\hat{Y}_t)\}dt, \quad Y_0 = x.$$

Owing to the estimates obtained in Theorem 2.1 we see that (4.6) has a unique solution and it satisfies

$$\bar{v}(0,x) = v_*(0,x) = E[\int_0^T \varphi(\hat{Y}_s, \hat{Z}_s)ds - \gamma\log v_0]$$

where $\hat{Z}_s = \hat{Z}(s,\hat{Y}_s)$.

Let us consider the following stochastic control problem with the averaging cost criterion

$$(4.7) \qquad \rho(\gamma) = \inf_{Z.\in\bar{\mathcal{A}}} \limsup_{T\to\infty} \frac{1}{T}E[\int_0^T \varphi(Y_s, Z_s)ds],$$

where Y_t is a controlled process governed by controlled stochastic differential equation (4.5) with control Z_t. The solution Y_t of (4.5) is sometimes written as Y_t^Z to make clear the dependence on the control Z_t. The set $\tilde{\mathcal{A}}$ of all admissible controls is defined as follows. Let $\bar{w}$ be the solution of H-J-B equation ergodic type (3.1). Then

$$\tilde{\mathcal{A}} = \{Z_\cdot; Z_t \text{ is an } R^{n+m} \text{ valued progressively measurable process such that } \limsup_{T\to\infty} \tfrac{1}{T}E[|Y_T^{(Z)}|^2] = 0\}$$

For this stochastic control problem there corresponds H-J-B equation of ergodic type (3.1) which can be written as

$$-\chi(\gamma) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{w}] + G^* D\bar{w} + \inf_{z\in R^{n+m}}\{(\lambda z)^* D\bar{w} + \varphi(x,z)\} \tag{4.8}$$

We then set

$$\hat{z}(x) = -N_\gamma^{-1}(\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})(x), \tag{4.9}$$

and consider stochastic differential equation

$$\begin{aligned} d\bar{Y}_t &= \lambda(\bar{Y}_t)dW_t + \{G(\bar{Y}_t) + \lambda(\bar{Y}_t)\hat{z}(\bar{Y}_t)\}dt, \\ &= \lambda(\bar{Y}_t)dW_t + \{\beta_\gamma - \lambda N_\gamma^{-1}\lambda^* D\bar{w}\}(\bar{Y}_t)dt \qquad (4.10) \\ \bar{Y}_0 &= x \end{aligned}$$

We shall prove

Proposition 4.1 $-\chi(\gamma) = \rho(\gamma)$ *and*

$$\rho(\gamma) = \lim_{T\to\infty}\frac{1}{T}E[\int_0^T \varphi(\bar{Y}_s,\bar{Z}_s)ds], \tag{4.11}$$

where $\bar{Z}_s = \hat{z}(\bar{Y}_s)$.

The following lemma plays important role in the proof of the above proposision and later discussions.

Lemma 4.2 *Under assumptions (2.11), (2.12),(2.20) and (3.5) the following estimates hold. There exists a positive constant* $\delta > 0$ *and* $C > 0$ *independent of* T *and* γ *with* $\gamma_1 \le \gamma \le \gamma_0$ *such that*

$$E[e^{\delta\bar{w}(\bar{Y}_T)}] \le C, \tag{4.12}$$

and also

$$E[e^{\delta|\bar{Y}_T|^2}] \le C. \tag{4.13}$$

Let us define

$$\bar{\chi}(\gamma) = \limsup_{T\to\infty}\frac{1}{T}\inf_{Z\in\tilde{\mathcal{A}}} E[\int_0^T \varphi(Y_s,Z_s)ds] = \limsup_{T\to\infty}\frac{1}{T}\bar{v}(0,x;T) \tag{4.14}$$

Then, we can see that

$$\bar{\chi} \le \rho(\gamma) = -\chi(\gamma).$$

Proposition 4.2 *Assume (2.11),(2.12),(2.20) and (2.22). Then,*

$$\bar{\chi}(\gamma) = \rho(\gamma) = -\chi(\gamma)$$

Proof of Theorem 2.2 is direct from this proposition since $\bar{\chi}(\gamma) = -\hat{\chi}(\gamma)$ because of Proposition 2.1.

The following is a direct consequence of proposition 4.1. Indeed,

Corollary 4.1 *Under the assumptions of Proposition 4.2 $\rho(\gamma)$ is a concave function on $(-\infty, 0)$ and $\hat{\chi}(\gamma)$ is a convex function.*

Indeed,

$$\varphi = \frac{1}{2}z^*z - \frac{\gamma}{2}z^*\sigma^*(\sigma\sigma^*)^{-1}\sigma - z^*\Sigma^*\hat{\alpha} + \frac{1}{2}\hat{\alpha}\Sigma\Sigma^*\hat{\alpha}$$

is a concave function of γ and so the infimum of a family of concave functions $\rho(\gamma)$ is concave.

Proposition 4.3 *Under the assumptions of proposition 3.1 $\bar{L}$ is ergodic.*

Proof.

$$\bar{L}\bar{w} = -\frac{1}{2}(D\bar{w})^*\lambda N_\gamma^{-1}\lambda^* D\bar{w} + \frac{\gamma}{2(1-\gamma)}\hat{\alpha}^*\Sigma\Sigma^*\hat{\alpha} - \chi \to -\infty$$

as $|x| \to \infty$ and $\bar{L}\bar{w} \le -c, \quad |x| >> 1,\ c > 0$. Moreover, $\bar{w}(x) \to \infty, \quad |x| \to \infty$ and Hasiminskii conditions hold.

$\square$

Remark 4.1 *The generator of the optimal diffusion process governed by (2.25) for risk-sensitive control problem (2.10) is defined by*

$$L_\infty\psi := \frac{1}{2}tr[\lambda\lambda^* D^2\psi] + [\beta_\gamma^* + \frac{\gamma}{1-\gamma}(Dw)^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^*]D\psi$$

On the other hand, in proving Theorem 2.2 we introduce another kind of stochastic control problem.

$$\rho(\gamma) = \inf_{Z_\cdot \in \bar{\mathcal{A}}} \limsup_{T\to\infty} \frac{1}{T}E[\int_0^T \varphi(Y_s, Z_s)ds],$$

where Y_t is a controlled process governed by stochastic differential equation

$$dY_t = \lambda(Y_t)dW_t + \{G(Y_t) + \lambda(Y_t)Z_t\}dt, \quad Y_0 = x$$

with controlled process Z_t, which is an R^{n+m} valued progressively measurable process. The generator of the optimal diffusion process for this problem is defined by

$$\begin{aligned}\bar{L}\psi &= \tfrac{1}{2}tr[\lambda\lambda^* D^2\psi] + (G + \lambda\hat{z})^* D\psi \\ &= \tfrac{1}{2}tr[\lambda\lambda^* D^2\psi] + [\beta_\gamma^* + (Dw)^*\lambda N_\gamma^{-1}\lambda^*]D\psi\end{aligned}$$

Here we note that $\bar{L}$ is related to L_∞ through the Gäuge transformation:

$$[e^{-w}L_\infty e^w]\varphi = [\bar{L} - (\gamma\eta - \chi(\gamma))]\varphi$$

and we see that ψ_∞ is an eigenfunction of $L_\infty + \gamma\eta$:

$$(L_\infty + \gamma\eta)\psi_\infty = \chi(\gamma)\psi_\infty$$

for the principal eigenvalue $\chi(\gamma)$ (cf. [6]).

5 Derived Poisson equation

We are going to consider Poisson equation formally obtained by differentiating H-J-B equation (3.1) of ergodic type with respect to γ. Namely, we consider

$$-\theta(\gamma) = \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + G^* Du - (\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})^* N_\gamma^{-1}\lambda^* Du$$
$$-\tfrac{1}{2(1-\gamma)^2}(\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})^*\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma(\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})$$

Since

$$-\tfrac{1}{2(1-\gamma)^2}(\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})^*\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma(\lambda^* D\bar{w} - \Sigma^*\hat{\alpha})$$
$$= -\tfrac{1}{2(1-\gamma)^2}(\sigma\lambda^* D\bar{w} - \hat{\alpha})^*(\sigma\sigma^*)^{-1}(\sigma\lambda^* D\bar{w} - \hat{\alpha})$$

we may write

$$-\theta(\gamma) = \bar{L}u - \frac{1}{2(1-\gamma)^2}(\sigma\lambda^* D\bar{w} - \hat{\alpha})^*(\sigma\sigma^*)^{-1}(\sigma\lambda^* D\bar{w} - \hat{\alpha}) \tag{5.1}$$

Note that $\bar{L}$ is ergodic in view of Proposition 4.3 and the pair $(u, \theta(\gamma))$ of a function u and a constant $\theta(\gamma)$ is considered the solution to (5.1). Let us set

$$\mathcal{D} = B_{R_0} = \{x \in R^n; |x| \le R_0\}$$

and R_0 is taken so large that

$$K(x;\bar{w}) := \frac{1}{2}(D\bar{w})^*\lambda N_\gamma^{-1}\lambda^* D\bar{w} - \frac{\gamma}{2(1-\gamma)}\hat{\alpha}^*\Sigma\Sigma^*\hat{\alpha} + \chi > 0, \quad x \in \mathcal{D}^c \tag{5.2}$$

for $\gamma \le \gamma_0 < 0$, which is possible because of assumption (2.22). Therefore, we see that $\bar{L}$, and $\bar{w}$ satisfy the assumption (7.3) in the last section. Thus according to Proposition 7.4 we can show existence of the solution $(u, \theta(\gamma))$ to (5.1).

Corollary 5.1 *(5.1) has a solution $(u, \theta(\gamma))$ such that*

$$\sup_{x\in\mathcal{D}^c}\frac{|u|}{\bar{w}} < \infty, \quad u \in W^{2,p}_{loc}$$

and

$$\theta(\gamma) = -\int \frac{1}{2(1-\gamma)^2}(\sigma\lambda^* D\bar{w} - \hat{\alpha})^*(\sigma\sigma^*)^{-1}(\sigma\lambda^* D\bar{w} - \hat{\alpha})m_\gamma(y)dy$$

Moreover, such solution u is unique up to additive constants.

Proof. It is obvious that

$$\frac{1}{2(1-\gamma)^2}(\sigma\lambda^* D\bar{w} - \hat{\alpha})^*(\sigma\sigma^*)^{-1}(\sigma\lambda^* D\bar{w} - \hat{\alpha}) \in F_K$$

and Proposition 7.4 applies.

6 Differentiability of H-J-B equation

Lemma 6.1 *Under the assumptions of Proposition 4.2*

$$\int e^{\delta|x|^2} m_\gamma(dx) \leq c, \tag{6.1}$$

where c and δ are positive constants independent of γ in $\gamma_1 \leq \gamma \leq \gamma_0 < 0$.

Proof. (6.1) is a direct consequence of (4.13) in Lemma 4.2 since $\bar{Y}_t$ is an ergodic diffusion process with the invariant measure $m_\gamma(dx)$.

□

In what follows we always asuume the assumptions of Theorem 2.2 (Propisition 4.2).

Lemma 6.2 *Let $(\bar{w}^{(\gamma)}, \chi(\gamma))$, $(\bar{w}^{(\gamma+\Delta)}, \chi(\gamma+\Delta))$ be solutions to (3.1) with γ, $\gamma+\Delta$ respectively such that $\bar{w}^{(\gamma)}(0) = 0$, and $\bar{w}^{(\gamma+\Delta)}(0) = 0$. Then $\bar{w}^{\gamma+\Delta}$ converges to $\bar{w}^{\gamma}$, H^1_{loc} strongly and unifromly for each compact set.*

Theorem 6.1 *Let $(\bar{w}^{(\gamma)}, \chi(\gamma))$, $(\bar{w}^{(\gamma+\Delta)}, \chi(\gamma+\Delta))$ be solutions to (3.1) with γ, $\gamma+\Delta$ respectively. Set $\chi^{(\Delta)} = \frac{\chi(\gamma+\Delta)-\chi(\gamma)}{\Delta}$ and $\zeta^{(\Delta)} = \frac{\bar{w}^{(\gamma+\Delta)}-\bar{w}^{(\gamma)}}{\Delta}$. Then,*

$$\lim_{|\Delta|\to 0} \chi^{(\Delta)} = \theta(\gamma) \tag{6.2}$$

and

$$\lim_{|\Delta|\to 0} \zeta^{(\Delta)}(x) = u(x)$$

where $(u, \theta(\gamma))$ is the solution to (5.1).

7 Appendix

Let L_0 be an elliptic operator defined by

$$L_0 u := \frac{1}{2}\sum_{i,j} a^{ij}(x) D_{ij} u + \sum_i b^i(x) D_i u \tag{7.1}$$

where $a^{i,j}(x)$ and $b^i(x)$ are Lipshitz continuous function such that

$$k_0|y|^2 \leq y^* a(x) y \leq k_1|y|^2, \quad \forall y \in R^N, \ k_0, \ k_1 > 0. \tag{7.2}$$

We assume that there exists a positive function $\psi \in C^2(R^N)$ such that

$$\begin{cases} \psi(x) \to \infty, \quad |x| \to \infty \\ -L_0\psi - \frac{c}{\psi}(D\psi)^* a D\psi \geq 0, \quad x \in B_R^c, \ \exists R > 0, c > 0 \\ L_0\psi < -1, \quad x \in B_R^c \end{cases} \tag{7.3}$$

Set $K(x;\psi) = -L_0\psi$,

$$F_\psi = \{u \in W^{2,p}_{loc}; \sup_{x\in B_R^c} \frac{|u(x)|}{\psi(x)} < \infty\}, \quad F_K = \{f \in W^{2,p}_{loc}; \sup_{x \in B_R^c} \frac{|f(x)|}{K(x;\psi)} < \infty\}$$

and

$$\mathcal{D} = B_R = \{x \in R^n; |x| \leq R\}.$$

Then, we consider the following exterior Dirichlet problem for a given bounded Borel function h on $\Gamma = \partial\mathcal{D}$:

$$\begin{cases} -L_0\xi = 0, & x \in \bar{\mathcal{D}}^c \\ \xi|_\Gamma = h \end{cases} \tag{7.4}$$

Proposition 7.1 *Exterior Dirichlet problem (7.4) has a unique bounded solution $\xi \in W^{2,p}_{loc} \cap L^\infty$.*

Let us take a bounded domain $\mathcal{D}_1$ such that $\mathcal{D} \subset \mathcal{D}_1$ and a bounded Borel function ϕ on $\Gamma_1 = \partial\mathcal{D}_1$. We consider a Dirichlet problem

$$\begin{cases} -L_0\zeta = 0 & \mathcal{D}_1 \\ \zeta|_{\Gamma_1} = \phi, \end{cases} \tag{7.5}$$

which admits a solution $\zeta \in W^{2,p}(\mathcal{D}_1) \cap L^\infty$, $\zeta - \phi \in W^{1,2}_0(\mathcal{D}_1)$. For this solution we consider an exterior Dirichlet problem (7.4) with $h = \zeta$. Then, we define an operator $P : \mathbf{B}(\Gamma_1) \mapsto \mathbf{B}(\Gamma_1)$ defined by

$$P\phi(x) = \xi(x), \quad x \in \Gamma_1,$$

where $\xi(x)$ is the solution to (7.4) with $h = \zeta$. Then, in a similar way to Lemma 5.1 in Chapter II in [1] we have

$$\sup_{B \in \mathcal{B}(\Gamma_1),\, x,y \in \Gamma_1} \lambda_{x,y}(B) < 1 \tag{7.6}$$

where

$$\lambda_{x,y}(B) = P\chi_B(x) - P\chi_B(y), \quad B \in \mathcal{B}(\Gamma_1)$$

Moreover, we have the following proposition (cf. Theorem 4.1, Chapter II in [1]).

Proposition 7.2 *The above defined P satisfies the following properties.*

$$\| P\phi \|_{L^\infty(\Gamma_1)} \leq \| \phi \|_{L^\infty(\Gamma_1)}, \quad P1(x) = 1 \tag{7.7}$$

and for some $\delta > 0$

$$P\chi_B(x) - P\chi_B(y) \leq 1 - \delta, \quad x, y \in \Gamma_1, \ B \in \mathcal{B}(\Gamma_1) \tag{7.8}$$

Furthermore, there exists a probability measure $\pi(dx)$ on $(\Gamma_1, \mathcal{B}(\Gamma_1))$ such that

$$|P^n\phi(x) - \int \phi(x)\pi(dx)| \leq K \| \phi \|_{L^\infty} e^{-\rho n}, \quad \rho = \log\frac{1}{1-\delta}, \ K = \frac{2}{1-\delta}, \tag{7.9}$$

and

$$\int \phi(x)\pi(dx) = \int P\phi(x)\pi(dx) \tag{7.10}$$

for all bounded Borel function ϕ.

Consider an exterior Dirichlet problem for a given function $f \in F_K$:

$$\begin{cases} -L_0 u = f, & x \in \mathcal{D}^c \\ u|_\Gamma = 0 \end{cases} \tag{7.11}$$

Then, we have the following Proposition.

Proposition 7.3 *For a given function $f \in F_K$ there exists a unique solution $u \in W^{2,p}_{loc}$ to (7.11) such that*

$$\sup_{x \in \mathcal{D}^c} \frac{|u(x)|}{\psi(x)} < \infty.$$

Let f be a function on R^n such that f is bounded in $\mathcal{D}$ and $f \in F_K(\mathcal{D}^c)$, and $\mathcal{D}_1$ a bounded domain such that $\mathcal{D} \subset \mathcal{D}_1$. We consider

$$\begin{cases} -L_0 \Psi = f & \mathcal{D}_1 \\ \Psi|_{\Gamma_1} = 0 \end{cases}$$

and

$$\begin{cases} -L_0 \xi = f & R^n \cap \overline{\mathcal{D}}^c \\ \xi|_\Gamma = \Psi_\Gamma \end{cases}$$

Then we set

$$Tf(x) = \xi(x), \quad x \in \Gamma_1$$

and

$$\nu(f) = \frac{\int_{\Gamma_1} Tf(\sigma)\pi(d\sigma)}{\int_{\Gamma_1} T1(\sigma)\pi(d\sigma)} \tag{7.12}$$

We further consider

$$\begin{cases} -L_0 z = f \\ z \in W^{2,p}_{loc}, \ \sup_{x \in \mathcal{D}^c} \frac{|z|}{\psi} < \infty \end{cases} \tag{7.13}$$

Then, in a similar way to the proof of Theorem 5.3, Chapter II in [1] we obtain the following proposition.

Proposition 7.4 *(7.13) has a solution unique up to additive constants if and only if $\nu(f) = 0$. Moreover*

$$\nu(f) = \int m(y) f(y) dy \tag{7.14}$$

for $m \in L^1(R^n)$, $m \geq 0$ and $-L_0^* m = 0$ in distribution sense:

$$\int m(y)(-L_0 z) dy = 0, \quad z \in W^{2,p}_{loc} \tag{7.15}$$

such that $z \in F_\psi(\mathcal{D}^c)$ and $-L_0 z \in F_K$. Furthermore $m(x)$ is the only function in L^1 satisfying (7.15) and

$$\int m(x) dx = 1.$$

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